Natural Resonant Frequencies of Microwave Dielectric Resonators

The application of dielectric resonators are currently of considerable interest in microwave techniques [1], [2]. The design of a dielectric resonator, as in the case of a metal cavity, depends on its natural resonant frequencies. Since exact solutions of dielectric resonators having shapes other than a sphere [3] or a doughnut [4] cannot be rigorously computed, approximate techniques must be adopted to solve the problem. Two types of resonant modes can be excited in a dielectric resonator, namely, the H mode and the E mode. The H mode is defined as the mode which has a large normal component of magnetic field at the boundary surfaces; and the E mode is the mode with no predominant normal component of magnetic field at the surfaces.

The electromagnetic fields in dielectric resonators of high permittivity approximately satisfy the open-circuit boundary condition. This can be verified by considering a plane wave propagating in the dielectric, incident on the boundary between dielectric and air [5]. The open-circuit boundary (OCB) condition is defined as

$$n \times H = 0$$
$$n \cdot E = 0$$

where n is the unit vector normal to the boundary. In other words, at the OCB, the normal component of electric field and the tangential component of magnetic field vanish. In fact, all E modes and some higher order H modes satisfy the OCB condition very well if the relative permittivity is much larger than unity. This can be seen by comparing a spherical dielectric resonator and a perfect OCB spherical resonator of the same size and the same permittivity. For a relative permittivity $\epsilon_r = 100$, the differences of resonant frequencies of the E modes is smaller than 1 per cent, the H_{2mn} (where the subscripts indicate the number of variations in the spherical coordinate r, θ, ϕ direction, respectively) modes differ approximately by 3 per cent. However, the difference is about 14 per cent for the H_{1mn} modes. Therefore, the resonant frequencies and the field distributions of dielectric resonators can be calculated, as an approximation, by assuming OCB condition; except some modifications are necessary for some lower order H modes.

Consider a homogeneous, lossless, circular cylindrical dielectric resonator of radius a and length L. The resonant frequencies are, approximately, given by the roots of the following equations [5]:

$$J_m'(\beta a) = 0, \quad \text{for } TE_{lmn} \text{ mode} \qquad (1)$$

$$J_m(\beta a) = 0, \quad \text{for } TM_{lmn} \text{ mode} \qquad (2)$$

where

$$\beta^2 = (2\pi/\lambda_0)^2[\epsilon' - (n\lambda_0/2L)^2] \qquad (3)$$

and λ_0 is the free space wavelength, J_m is the mth order Bessel's function of the first kind.

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TABLE I

THEORETICAL AND EXPERIMENTAL RESONANT FREQUENCIES OF CYLINDRICAL $SrTiO_3$ RESONATORS

a (mils)	L (mils)	Mode*	f_0(GC) (Theoretical) OCB approximation	f_0(GC) (Theoretical) modified OCB approximation	f(GC) measured
32.8	68	$TE_{10\delta}$	8 26	9 18	9.28
		$TE_{11\delta}$	13 15	13.9	13.827
		TM_{101}	14 0	/	13.66
32.25	55.3	$TE_{10\delta}$	8.39	9.45	9.71
		$TE_{11\delta}$	13.36	14.15	14.24
		TM_{101}	14.5	/	14.203
35.2	30.6	$TE_{10\delta}$	7 68	10 27	10 43
		$TE_{11\delta}$	12 24	14 46	14 335
		TM_{101}	16.10	/	16.23

* $TE_{10\delta}$ and $TE_{11\delta}$ are H modes, TM_{101} is E mode.

The subscripts, l, m, n, denote the number of variations of the fields in the ρ, ϕ, z-direction respectively. Note that the TE_{lm0} and the TM_{1mn} modes need some modifications.

To obtain a better solution for the resonant frequencies of the TE_{lm0} modes, it is assumed that only the circular surface satisfies the OCB condition. The ϕ-component of electric field inside the resonator is then

$$E_\phi{}^i = A_i(z)J_m(\beta\rho)\cos m\phi \qquad (4)$$

The field immediately outside the resonator has the same distribution as that of the field just inside the resonator. Hence, the electric field just outside the resonator may be expressed as

$$E_\phi{}^0 = A_0(z)J_m(\beta\rho)\cos m\phi \qquad (5)$$

Substituting (4) and (5) into Maxwell's equations and matching the solutions at $z = \pm L/2$ yield

$$\zeta_0 = \zeta \tan(\zeta L/2) \qquad (6)$$

$$\zeta^2 + \epsilon_r\zeta_0{}^2 = (\epsilon_r - 1)\beta^2 \qquad (7)$$

where

$$\zeta^2 = (2\pi/\lambda_0)^2\epsilon_r - \beta^2 \qquad (8)$$

$$\zeta_0{}^2 = \beta^2 - (2\pi/\lambda_0)^2 \qquad (9)$$

and β is given by (1). The two simultaneous equations (6) and (7) can be solved for ζ and ζ_0 graphically or by a computer. With the knowledge of ζ or ζ_0, the resonant frequency can be computed by (8) or (9). Physically, the resonant mode is no longer the TE_{lm0} mode, since the field has a fraction δ of one-half cycle sinusoidal variation along the z direction [1], where $\delta = L\zeta/\pi$. A similar modification for TM_{1mn} mode can be made by assuming that only two flat surfaces satisfy the OCB condition. The resonant frequencies are approximately given by [5]

$$\frac{J_m'(\beta a)}{J_m(\beta a)} + \frac{\beta}{\epsilon_r\alpha}\frac{K_m'(\alpha a)}{K_m(\alpha a)} = \pi^2(kL\beta a)^{-2}\frac{J_m'(\beta a)}{J_m''(\beta a)} + \beta^3\alpha^{-3}\frac{K_m'(\alpha a)}{K_m''(\alpha a)} \qquad (10)$$

where

$$\alpha^2 = (n\pi/L)^2 - (\pi/\lambda_0)^2$$
$$\beta^2 = (2\pi/\lambda_0)^2\epsilon_r - (n\pi/L)^2$$
$$k^2 = (2\pi/\lambda_0)^2\epsilon_r$$

K_m is the modified Bessel's function of the second kind. Equation (10) can be solved for λ_0 by computer. The method outlined above is applicable to rectangular and elliptic cylindrical dielectric resonators.

Measured results made on cylindrical $SrTiO_3$ ($\epsilon_r = 279$) resonators agree well with the theoretical values. They are listed in Table I.

ACKNOWLEDGMENT

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A Note on Varactor Frequency Tripler

Several papers [1]–[9] in the past ten years have considered analysis of harmonic generators with a nonlinear reactance. The most comprehensive work on the varactor multipliers is by Penfield and Rafuse [10], in which a detailed analysis of the frequency tripler has been made using abrupt-junction varactors. Leonard [9] has made a detailed analysis of varactor frequency doublers to predict the power and efficiency for nonlinearities ranging from graded-junction to hyperabrupt-junction varactors.

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This correspondence is concerned with the detailed analysis of the varactor frequency tripler to determine conditions of optimum input and output impedances, maximum efficiency, and power with the variable exponent $\gamma(\frac{1}{3}<\gamma<\frac{3}{4})$ which characterizes varactor nonlinearity. To achieve the best frequency tripler performance, idler current at the second harmonic is allowed to flow in the varactor. The basic varactor frequency tripler circuit is shown in Fig. 1. The analysis in this discussion assumes the varactor to be imbedded in a lossless coupling network.

The input and the output impedances and maximum efficiency as a function of γ are determined from the calculated values of the varactor voltages and currents. For frequency multipliers using a varactor, the voltage-charge relationship of the nonlinear capacitor can be expressed.

$$\left[\frac{q+q_\phi}{Q_B+q_\phi}\right]=\left[\frac{V+\phi}{V_B+\phi}\right]^{1-\gamma}\quad(\gamma\neq 1)\qquad(1)$$

Where

V_B = The varactor breakdown voltage
ϕ = The varactor electrostatic potential
q_ϕ = The charge at the varactor electrostatic potential

If current flow through the diode is constrained by the coupling network to include only the fundamental second and third harmonic frequencies, the normalized charge can be expressed as

$$\begin{aligned}\hat{q}&=\frac{q+q_\phi}{Q_B+q_\phi}\\&=\hat{q}_0+\hat{q}_1\exp(j\omega t)+\hat{q}_1^*\exp(-j\omega t)\\&\quad+\hat{q}_2\exp(j2\omega t)+\hat{q}_2^*\exp(-j2\omega t)\\&\quad+\hat{q}_3\exp(j3\omega t)+\hat{q}_3^*\exp(-j3\omega t)\qquad(2)\end{aligned}$$

Consequently, real and reactive power at each of these frequencies can be determined for a given varactor. The power at second harmonic is assumed to appear totally reactive in lossless coupling network of the circuit. According to the Manley-Rowe condition [11], the sum of the average power in the varactor at the fundamental frequency and third harmonic must be equal to zero, so the nonlinear element serves as a power sink at the fundamental frequency and as a power source at the third harmonic. The circuit of Fig. 1 illustrates the condition of the power transfer from the fundamental to the third harmonic.

$$P_{\text{ave}^1}=-P_{\text{ave}^3}\qquad(3)$$

where

P_{ave^1} = Average power at fundamental frequency
P_{ave^3} = Average power at third harmonic

Breitzer et al. [5], [12] have given the coefficients of the normalized charge expansion $\hat{q}$ for the abrupt-junction varactor $(\gamma=\frac{1}{2})$ operated in the region between breakdown voltage to forward-biased conduction containing no harmonics higher than third harmonic. The normalized charge harmonic expansion $\hat{q}$ is

$$\begin{aligned}\hat{q}&=0.5+0.310\sin(\omega t)+0.168\sin(2\omega t)\\&\quad+0.155\sin(3\omega t)\qquad(4)\end{aligned}$$

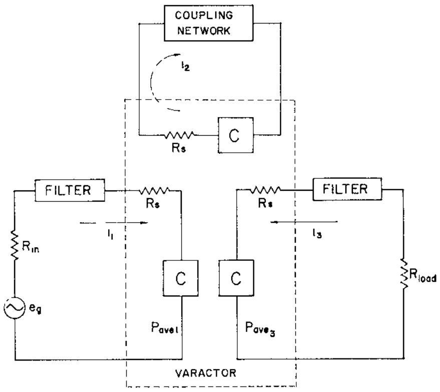

Fig. 1. Equivalent circuit of a varactor frequency tripler.

which we assume to be valid for any arbitrary value of γ. The maximum power converted by a varactor as a function of γ can be obtained using (2), (3), and (4), and is plotted in Fig. 2.

$$P_{\text{ave}}=A(\gamma)P_{\text{norm}}\frac{\omega}{\omega_c}\qquad(5)$$

where

$$\begin{aligned}A(1/3)&=0.0104\\A(1/2)&=0.0242\\A(2/3)&=0.0545\\A(3/4)&=0.0830\end{aligned}$$

$$P_{\text{norm}}=\frac{(V_B+\phi)^2}{R_s}=\text{normalization power}$$

The current through the varactor in terms of γ can be written

$$\begin{aligned}i&=\frac{0.310}{1-\gamma}\omega C_{\min}(V_B+\phi)\sin(\omega t)\\&\quad+\frac{0.336}{1-\gamma}\omega C_{\min}(V_B+\phi)\sin(2\omega t)\\&\quad+\frac{0.465}{1-\gamma}\omega C_{\min}(V_B+\phi)\sin(3\omega t)\qquad(6)\end{aligned}$$

Considering the input, output, and idler coupling circuits lossless, the input and load resistance of Fig. 1 become

$$\begin{aligned}R_{\text{in}}&=R_s\left[1+M(\gamma)\frac{\omega_c}{\omega}\right]\\&\simeq M(\gamma)R_s\frac{\omega_c}{\omega}\qquad\text{for }\omega<10^{-2}\omega_c\qquad(7)\end{aligned}$$

and

$$\begin{aligned}R_{\text{load}}&=R_s\left[N(\gamma)\frac{\omega_c}{\omega}-1\right]\\&\simeq N(\gamma)R_s\frac{\omega_c}{\omega}\qquad\text{for }\omega<10^{-2}\omega_c\qquad(8)\end{aligned}$$

where

$$M(\gamma)=\frac{(1-\gamma)^2}{0.04815}A(\gamma)$$

$$N(\gamma)=\frac{(1-\gamma)^2}{0.10806}A(\gamma)$$

Figure 3 shows the coefficiencies of $M(\gamma)$ and $N(\gamma)$ relationship as a function of γ. The conversion efficiency can now be calculated

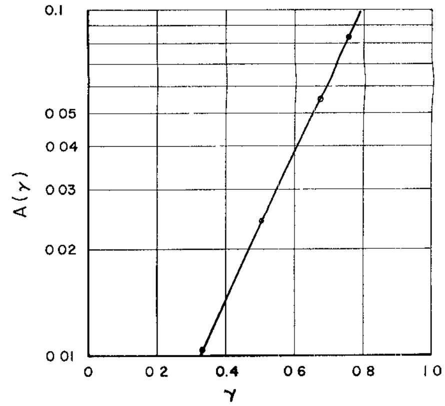

Fig. 2. $A(\gamma)$ plotted as a function of γ.

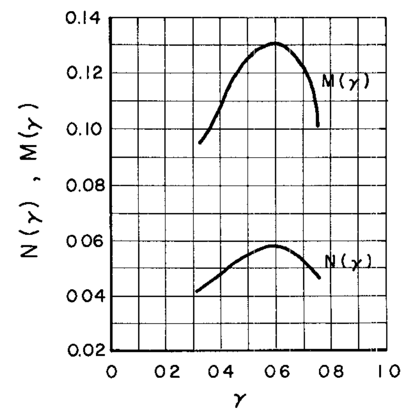

Fig. 3. Plot of $M(\gamma)$ and $N(\gamma)$ as a function of γ.

from the ratio of output power to the input power and considerable algebraic reduction.

$$\eta=\frac{P_{\text{out}}}{P_{\text{in}}}$$

$$=1-(a+b)\frac{\omega}{\omega_c}+\frac{a(a+b)\left(\frac{\omega}{\omega_c}\right)^2}{1+a\frac{\omega}{\omega_c}}\qquad(9)$$

where

$$a=\frac{0.10013}{(1-\gamma)^2A(\gamma)}$$

and

$$b=\frac{0.10811}{(1-\gamma)^2A(\gamma)}$$

For $\omega<10^{-2}\omega_c$, the approximate conversion efficiency can be written as:

$$\eta=1-h(\gamma)\frac{\omega}{\omega_c}\qquad(10)$$

where

$$h(\gamma)=\frac{0.20824}{(1-\gamma)^2A(\gamma)}\qquad(10)$$

Figure 4 shows a plot of $h(\gamma)$ vs. γ. It is clear from the plot that the conversion efficiency will be highest where γ is in the neighborhood of 0.6 and is not much reduced where γ lies in the range 0.5 to 0.7. Figure 5 is a plot

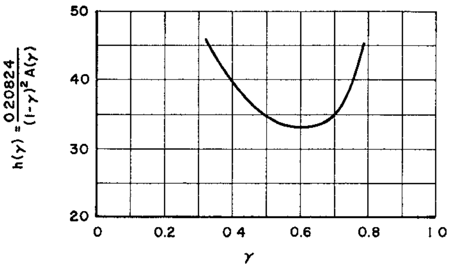

Fig. 4. $h(\gamma)$ as a function of γ.

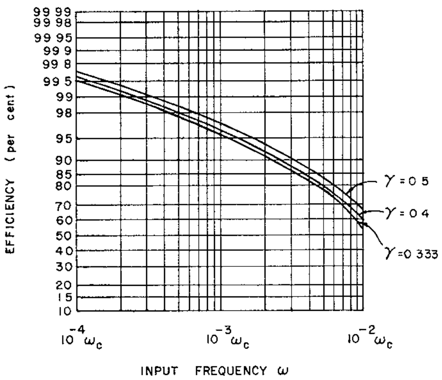

Fig. 5. Efficiency as a function of ω for several values of γ.

of the conversion efficiency as a function of input frequency ω for several values of γ. The conversion efficiency varies linearly with frequency and is quite near 100 per cent at low frequencies.

Conclusions

The results given in this analysis indicate that the highest efficiency would occur with a varactor nonlinearity γ lying in the range 0.5 to 0.7 assuming the same cutoff frequency for all values of γ. The power handling capability with γ's in the range $0.6<\gamma<1$ is much higher than that in the range $0<\gamma<0.6$ with the same conversion efficiency. However, the power handling capability of varactors has been shown to depend strongly on breakdown voltage and series resistance R_s.

The above equations can be extended to a cascade of multiplier stages considering that the load impedance of each stage is the input impedance of the following stage and that the output power of each stage is the input power of the following stage.

Acknowledgment

The author wishes to thank T. Leonard and P. Penfield for valuable suggestions and encouragement in this work.

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Corrections

Intrinsic Attenuation

In the above paper,[1] on page 180, (4) should have read:

$$\Gamma_{TM}=\frac{B}{2A}\left[1\pm\sqrt{1-\left(\frac{2\,|A|}{B}\right)^2}\right].$$

It will then be in agreement with (2) of a previous correspondence item,[2] where it appeared correctly.

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[1] Beatty, R. W., Intrinsic attenuation, *IEEE Trans. on Microwave Theory and Techniques*, vol MTT-11, May 1963, pp 179–182.

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Electromagnetic Wave Propagation in Lossy Ferrites

In the above paper,[1] the following corrections should be made:

1) Equation (6) should read:

$$\omega_C^2(\rho^2-\rho_0^2)^2+\omega_M\omega_C(\rho^2-\rho_0^2)(\rho^2-2k_0^2)-\omega^2(\rho^2-\rho_0^2)^2+\omega_M^2k_0^2(k_0^2-\rho^2)=0$$

2) Page 518, second column, first paragraph, first line reads: $\omega_C=\omega_M+j\omega_R$, should read: $\omega_C=\omega_H+j\omega_R$.

3) Page 519, first column, second paragraph, first line reads: "radial" should read "radical."

4) Equation (18b), coefficient of last term reads:

$$\left(\frac{\omega\rho_0k_0}{\rho_0^2\omega_{H_1}+k_0^2\omega_M}\right)^2$$

should read:

$$\left(\frac{\omega\rho_0k_0}{\rho_0^2\omega_{H_2}+k_0^2\omega_M}\right)^2$$

5) Page 523, second column, first paragraph, third line reads: $(\omega_M,\ \omega_R=0)$; should read: $(\omega_H,\ \omega_R=0)$.

6) Equation (26a), reads:

$$\left|\frac{P}{P_0}\right|=\frac{4\beta_0\epsilon_f\bar{\rho}'}{(\beta_0\epsilon_f)^2+|\rho|^2+2\beta_0\bar{\rho}'\epsilon_f}$$

should read:

$$\left|\frac{P}{P_0}\right|=\frac{4\beta_0\epsilon_f\rho}{(\beta_0\epsilon_f)^2+|\rho|^2+2\beta_0\rho'\epsilon_f}$$

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[1] Rosenbaum, F. J., *IEEE Trans. on Microwave Theory and Techniques*, vol MTT-12, Sep 1964, pp 517–528.

Internal Reflection in Dielectric Prisms

In the above,[1] on page 584, the membership status of the authors are incorrect. Dr. Fellers is a Fellow and Dr. Taylor is a Senior Member of IEEE.

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[1] Fellers, R. G., and J. Taylor, Internal reflection in dielectric prisms, *IEEE Trans. on Microwave Theory and Techniques*, vol MTT-12, Nov 1964, pp 584–587.